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Constructing Quantum Error-Correcting Codes for p^m-State Systems from Classical Error-Correcting Codes*

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SUMMARY We generalize the construction of quantum error-correcting codes from $\mathbf{F}_4$-linear codes by Calderbank et al. to p^m-state systems. Then we show how to determine the error from a syndrome. Finally we discuss a systematic construction of quantum codes with efficient decoding algorithms.

key words: *quantum codes, stabilizer codes, additive codes, self-orthogonal codes*

1. Introduction

Quantum error-correcting codes have attracted much attention. Among many research articles, the most general and systematic construction is the so called *stabilizer code construction* [6] or *additive code construction* [2], which constructs a quantum error-correcting code as an eigenspace of an Abelian subgroup S of the error group. Thereafter Calderbank et al. [3] proposed a construction of S from an additive code over the finite field $\mathbf{F}_4$ with 4 elements.

These constructions work for tensor products of 2-state quantum systems. However Knill [8], [9] and Rains [13] observed that the construction [2], [6] can be generalized to n-state systems by an appropriate choice of the error basis. Rains [13] also generalized the construction [3] using additive codes over $\mathbf{F}_4$ to p-state quantum systems, but his generalization does not relate the problem of quantum code construction to classical error-correcting codes. We propose a construction of quantum error-correcting codes for p^m-state systems from classical error-correcting codes which is a generalization of [3].

Throughout this paper, p denotes a prime number and m a positive integer. This paper is organized as follows. In Sect. 2, we review the construction of quantum codes for nonbinary systems. In Sect. 3, we propose a construction of quantum codes for p-state systems from classical codes over $\mathbf{F}_{p^2}$. In Sect. 4, we propose a construction of quantum codes for p^m-state systems from classical linear codes over $\mathbf{F}_{p^{2m}}$. In Sect. 5, we discuss a systematic construction of quantum codes with efficient decoding algorithms.

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2. Stabilizer Coding for p^m-State Systems

2.1 Code Construction

We review the generalization [8], [9], [13] of the construction [2], [6]. First we consider p-state systems. We shall construct a quantum code Q encoding quantum information in p^k-dimensional linear space into $\mathbb{C}^{p^n}$. Q is said to have minimum distance d and said to be an $[[n,k,d]]_p$ quantum code if it can detect up to $d-1$ quantum local errors. Let λ be a primitive p-th root of unity, C_p, D_λ $p\times p$ unitary matrices defined by $(C_p)_{ij} = \delta_{j-1,i \bmod p}$, $(D_\lambda)_{ij} = \lambda^{i-1}\delta_{i,j}$. Notice that C_2 and D_{-1} are the Pauli spin matrices σ_x and σ_z. We consider the error group E consisting of $\lambda^j w_1 \otimes \cdots \otimes w_n$, where j is an integer, w_i is $C_p^a D_\lambda^b$ with some integers a, b.

For row vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n)$, $(\boldsymbol{a}|\boldsymbol{b})$ denotes the concatenated vector $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ as used in [3]. For vectors $(\boldsymbol{a}|\boldsymbol{b}), (\boldsymbol{a}'|\boldsymbol{b}') \in \mathbf{F}_p^{2n}$, we define the alternating inner product

$$((\boldsymbol{a}|\boldsymbol{b}), (\boldsymbol{a}'|\boldsymbol{b}')) = \langle \boldsymbol{a}, \boldsymbol{b}' \rangle - \langle \boldsymbol{a}', \boldsymbol{b} \rangle, \tag{1}$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbf{F}_p^n$. For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbf{F}_p^n$, we define

$$X(\boldsymbol{a}) = C_p^{a_1} \otimes \cdots \otimes C_p^{a_n},$$

$$Z(\boldsymbol{a}) = D_\lambda^{a_1} \otimes \cdots \otimes D_\lambda^{a_n}.$$

Then we have

$$\begin{aligned} &X(\boldsymbol{a})Z(\boldsymbol{b})X(\boldsymbol{a}')Z(\boldsymbol{b}') \\ &\quad = \lambda^{\langle \boldsymbol{a},\boldsymbol{b}'\rangle - \langle \boldsymbol{a}',\boldsymbol{b}\rangle} X(\boldsymbol{a}')Z(\boldsymbol{b}')X(\boldsymbol{a})Z(\boldsymbol{b}). \end{aligned} \tag{2}$$

For $(\boldsymbol{a}|\boldsymbol{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbf{F}_p^{2n}$, we define the weight of $(\boldsymbol{a}|\boldsymbol{b})$ to be

$$\sharp\{i \mid a_i \neq 0 \text{ or } b_i \neq 0\}. \tag{3}$$

Theorem 1: Let C be an $(n-k)$-dimensional $\mathbf{F}_p$-linear subspace of $\mathbf{F}_p^{2n}$ with the basis $\{(\boldsymbol{a}_1|\boldsymbol{b}_1), \ldots, (\boldsymbol{a}_{n-k}|\boldsymbol{b}_{n-k})\}$, $C^\perp$ the orthogonal space of C with respect to the inner product (1). Suppose that $C \subseteq C^\perp$ and the minimum weight (3) of $C^\perp \setminus C$ is d. Then the subgroup S of E generated by $\{X(\boldsymbol{a}_1)Z(\boldsymbol{b}_1), \ldots, X(\boldsymbol{a}_{n-k})Z(\boldsymbol{b}_{n-k})\}$ is Abelian, and an eigenspace of S is an $[[n,k,d]]_p$ quantum code.

Next we consider quantum codes for p^m-state systems, where m is a positive integer. But the code construction for p^m-state systems is almost the same as that for p-state systems, because the state space of a p^m-state system can be regarded as the m-fold tensor products of that of a p-state system. We shall construct a quantum code encoding quantum information in p^{mk}-dimensional linear space into $\mathbb{C}^{p^{mn}}$. For $(\boldsymbol{a}|\boldsymbol{b}) = (a_{1,1}, a_{1,2}, \ldots, a_{1,m}, a_{2,1}, \ldots, a_{n,m}, b_{1,1}, \ldots, b_{n,m}) \in \mathbf{F}_p^{2mn}$, we define the weight of $(\boldsymbol{a}|\boldsymbol{b})$ to be

$$\sharp\{i \mid \text{there exists nonzero element in } \{a_{i,1}, \ldots, a_{i,m}, b_{i,1}, \ldots, b_{i,m}\}\}. \tag{4}$$

Corollary 2: Let C be an $(mn-mk)$-dimensional $\mathbf{F}_p$-linear subspace of $\mathbf{F}_p^{2mn}$ with the basis $\{(\boldsymbol{a}_1|\boldsymbol{b}_1), \ldots, (\boldsymbol{a}_{mn-mk}|\boldsymbol{b}_{mn-mk})\}$, $C^\perp$ the orthogonal space of C with respect to the inner product (1). Suppose that $C \subseteq C^\perp$ and the minimum weight (4) of $C^\perp \setminus C$ is d. Then the subgroup S of E generated by $\{X(\boldsymbol{a}_1)Z(\boldsymbol{b}_1), \ldots, X(\boldsymbol{a}_{mn-mk})Z(\boldsymbol{b}_{mn-mk})\}$ is Abelian, and an eigenspace of S is an $[[n,k,d]]_{p^m}$ quantum code.

2.2 Error Correction Procedure

In this subsection we review the process of correcting errors. Let $H = \mathbb{C}^p$, $H^{\otimes n} \supset Q$ the quantum code constructed by Theorem 1, and H_{env} the Hilbert space representing the environment. Suppose that we send a codeword $|\psi\rangle \in Q$, that the state of the environment is initially $|\psi_{\mathrm{env}}\rangle \in H_{\mathrm{env}}$, and that we receive $|\psi'\rangle \in H^{\otimes n} \otimes H_{\mathrm{env}}$. Then there exists a unitary operator U such that

$$|\psi'\rangle = U(|\psi\rangle \otimes |\psi_{\mathrm{env}}\rangle).$$

If U acts nontrivially τ $(0 \le \tau \le n)$ subsystems among n tensor product space $H^{\otimes n}$, then τ is said to be *the number of errors.*

We assume that $2\tau + 1 \le d$, where d is as in Theorem 1. If we measure each observable in $H^{\otimes n}$ whose eigenspaces are the same as those of $X(\boldsymbol{a}_i)Z(\boldsymbol{b}_i)$ for $i = 1, \ldots, n-k$, where $X(\boldsymbol{a}_i)Z(\boldsymbol{b}_i)$ is as defined in Theorem 1, then the entangled state $|\psi'\rangle$ is projected to $A|\psi\rangle \otimes |\psi'_{\mathrm{env}}\rangle$, for some $A \in E$ and $|\psi'_{\mathrm{env}}\rangle \in H_{\mathrm{env}}$, by the measurements. By the measurement outcomes we can find a unitary operator $A' \in E$ such that $A'A|\psi\rangle = |\psi\rangle$.

The determination of A' requires exhaustive search in general. Thus the computational cost finding A' from the measurement outcomes is large when both n and d are large. However, in certain special cases we can efficiently determine A'. An efficient method finding A' is presented in Sect. 3.2.

Remark 3: The error correction method presented in this subsection is not explicitly mentioned in the papers [2], [3], [6]. Still, it can be derived from general facts on quantum error correction presented in [1], [5], [10]. A readable exposition on the error correction procedure is provided by Preskill [16].

3. Construction of Quantum Codes for p-State Systems from Classical Codes

3.1 Codes for p-State Systems

In this subsection we describe how to construct quantum codes for p-state systems from additive codes over $\mathbf{F}_{p^2}$. Let ω be a primitive element in $\mathbf{F}_{p^2}$.

Lemma 4: $\{\omega, \omega^p\}$ is a basis of $\mathbf{F}_{p^2}$ over $\mathbf{F}_p$.

Proof: When $p = 2$ the assertion is obvious. We assume that $p \ge 3$. Suppose that $\omega^p = a\omega$ for some $a \in \mathbf{F}_p$. Then $\omega = \omega^{p^2} = (a\omega)^p = a^2\omega$, and a is either 1 or -1. If $a = 1$, then $\omega \in \mathbf{F}_p$ and ω is not a primitive element. If $a = -1$, then $\omega^{2p} = \omega^2$. This is a contradiction, because ω is a primitive element and $2p \not\equiv 2 \pmod{p^2-1}$. □

For $(\boldsymbol{a}|\boldsymbol{b}) \in \mathbf{F}_p^{2n}$ we define $\phi(\boldsymbol{a}|\boldsymbol{b}) = \omega\boldsymbol{a} + \omega^p\boldsymbol{b}$. Then the weight (3) of $(\boldsymbol{a}|\boldsymbol{b})$ is equal to the Hamming weight of $\phi(\boldsymbol{a}|\boldsymbol{b})$. For $\boldsymbol{c} = (c_1, \ldots, c_n), \boldsymbol{d} \in \mathbf{F}_{p^2}^n$, we define the inner product† of $\boldsymbol{c}$ and $\boldsymbol{d}$ by

$$\langle \boldsymbol{c}, \boldsymbol{d}^p\rangle - \langle \boldsymbol{c}^p, \boldsymbol{d}\rangle = \langle \boldsymbol{c}, \boldsymbol{d}^p\rangle - \langle \boldsymbol{c}, \boldsymbol{d}^p\rangle^p \tag{5}$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbf{F}_{p^2}^n$ and $\boldsymbol{c}^p = (c_1^p, \ldots, c_n^p)$. For $(\boldsymbol{a}|\boldsymbol{b}), (\boldsymbol{a}'|\boldsymbol{b}') \in \mathbf{F}_p^{2n}$ the inner product (5) of $\phi(\boldsymbol{a}|\boldsymbol{b})$ and $\phi(\boldsymbol{a}'|\boldsymbol{b}')$ is

$$\begin{aligned}
&\langle \phi(\boldsymbol{a}|\boldsymbol{b}), \phi(\boldsymbol{a}'|\boldsymbol{b}')^p\rangle - \langle \phi(\boldsymbol{a}|\boldsymbol{b})^p, \phi(\boldsymbol{a}'|\boldsymbol{b}')\rangle \\
&\quad = \langle \omega\boldsymbol{a} + \omega^p\boldsymbol{b}, \omega^p\boldsymbol{a}'^p + \omega\boldsymbol{b}'^p\rangle \\
&-\langle \omega^p\boldsymbol{a}^p + \omega\boldsymbol{b}^p, \omega\boldsymbol{a}' + \omega^p\boldsymbol{b}'\rangle \\
&\quad = (\omega^2 - \omega^{2p})(\langle \boldsymbol{a}, \boldsymbol{b}'\rangle - \langle \boldsymbol{a}', \boldsymbol{b}\rangle).
\end{aligned}$$

Since ω is a primitive element, $\omega^2 \ne \omega^{2p}$. Thus the inner product (1) of $(\boldsymbol{a}|\boldsymbol{b})$ and $(\boldsymbol{a}'|\boldsymbol{b}')$ is zero iff the inner product (5) of $\phi(\boldsymbol{a}|\boldsymbol{b})$ and $\phi(\boldsymbol{a}'|\boldsymbol{b}')$ is zero. Thus we have

Theorem 5: Let C be an additive subgroup of $\mathbf{F}_{p^2}^n$ containing p^{n-k} elements, C' its orthogonal space with respect to the inner product (5). Suppose that $C' \supseteq C$ and the minimum Hamming weight of $C' \setminus C$ is d. By identifying $\phi^{-1}(C)$ with an Abelian subgroup of E via $X(\cdot)Z(\cdot)$, any eigenspace of $\phi^{-1}(C)$ is an $[[n,k,d]]_p$ quantum code.

†The map (5) is $\mathbf{F}_p$-bilinear but does not take values in $\mathbf{F}_p$. It is neither $\mathbf{F}_{p^2}$-bilinear nor $\mathbf{F}_{p^2}$-sesquilinear. Thus calling the map (5) "inner product" is a little abusive. But the map (5) can be converted to an $\mathbf{F}_p$-bilinear form by dividing it by $\omega^2 - \omega^{2p}$. For this reason we call the map (5) "inner product."

We next clarify the self-orthogonality of a linear code over $\mathbf{F}_{p^2}$ with respect to (5).

Lemma 6: Let C be a linear code over $\mathbf{F}_{p^2}$, and C' the orthogonal space of C with respect to (5). We define $C^p = \{\boldsymbol{x}^p \mid \boldsymbol{x} \in C\}$ and $(C^p)^\perp$ the orthogonal space of C^p with respect to the standard inner product. Then we have $C' = (C^p)^\perp$.

Proof: It is clear that $C' \supseteq (C^p)^\perp$. Suppose that $\boldsymbol{x} \in C'$. Then for all $\boldsymbol{y} \in C$, $\langle \boldsymbol{x}, \boldsymbol{y}^p \rangle - \langle \boldsymbol{x}, \boldsymbol{y}^p \rangle^p = 0$. Thus $\langle \boldsymbol{x}, \boldsymbol{y}^p \rangle \in \mathbf{F}_p$. Since $\langle \boldsymbol{x}, \omega^p \boldsymbol{y}^p \rangle - \langle \boldsymbol{x}, \omega^p \boldsymbol{y}^p \rangle^p = 0$, $\omega^p \langle \boldsymbol{x}, \boldsymbol{y}^p \rangle \in \mathbf{F}_p$. Since $\omega^p \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p$, we conclude that $\langle \boldsymbol{x}, \boldsymbol{y}^p \rangle = 0$. □

Theorem 7: Let C be an $[n, (n-k)/2]$ linear code over $\mathbf{F}_{p^2}$ such that $C \subseteq (C^p)^\perp$. Suppose that the minimum Hamming weight of $(C^p)^\perp \setminus C$ is d. Then any eigenspace of $\phi^{-1}(C)$ is an $[[n,k,d]]_p$ quantum code.

3.2 Error Correction for p-State Systems

In this subsection we consider how to determine the error from measurements with quantum codes obtained via Theorem 7. We retain notations from Theorem 7. Suppose that $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r$ is an $\mathbf{F}_{p^2}$-basis of C. Then $\mathbf{F}_p$-basis of $\phi^{-1}(C)$ is $(\boldsymbol{a}_1|\boldsymbol{b}_1) = \phi^{-1}(\boldsymbol{g}_1), (\boldsymbol{a}_2|\boldsymbol{b}_2) = \phi^{-1}(\omega \boldsymbol{g}_1), \ldots, (\boldsymbol{a}_{2r}|\boldsymbol{b}_{2r}) = \phi^{-1}(\omega \boldsymbol{g}_r)$. Suppose that by the procedure in Sect. 2.2, the error is converted to $A \in E$ that corresponds to $\phi^{-1}(\boldsymbol{e})$ for some $\boldsymbol{e} \in \mathbf{F}_{p^2}^n$ via $X(\cdot)Z(\cdot)$, and the original quantum state is $|\psi\rangle$. We know which eigenspace of $X(\boldsymbol{a}_i)Z(\boldsymbol{b}_i)$ the vector $A|\psi\rangle$ belongs to. By Eq. (2)

$$X(\boldsymbol{a}_i)Z(\boldsymbol{b}_i)A|\psi\rangle = \lambda^\ell A|\psi\rangle,$$

where ℓ is the alternating inner product (1) of $(\boldsymbol{a}_i|\boldsymbol{b}_i)$ and $\phi^{-1}(\boldsymbol{e})$, which is denoted by $s_i \in \mathbf{F}_p$. Then we have

$$\langle \boldsymbol{g}_i, \boldsymbol{e}^p \rangle - \langle \boldsymbol{g}_i^p, \boldsymbol{e} \rangle = (\omega^2 - \omega^{2p}) s_{2i-1},$$

$$\langle \omega \boldsymbol{g}_i, \boldsymbol{e}^p \rangle - \langle \omega^p \boldsymbol{g}_i^p, \boldsymbol{e} \rangle = (\omega^2 - \omega^{2p}) s_{2i}.$$

It follows that $\langle \boldsymbol{g}_i^p, \boldsymbol{e} \rangle = (\omega^2 - \omega^{2p})(\omega s_{2i-1} - s_{2i})/(\omega^p - \omega)$. $\{\boldsymbol{g}_1^p, \ldots, \boldsymbol{g}_r^p\}$ can be used as rows of the check matrix of $(C^p)^\perp$. If we have a classical decoding algorithm for $(C^p)^\perp$ finding the error $\boldsymbol{e}$ from a classical syndrome $\langle \boldsymbol{g}_1^p, \boldsymbol{e} \rangle, \ldots, \langle \boldsymbol{g}_r^p, \boldsymbol{e} \rangle$, then we can find the quantum error $A \in E$.

Remark 8: In this section we assumed that ω is a primitive element in $\mathbf{F}_{p^2}$. It is enough to assume that ω belongs to $\mathbf{F}_{p^2}$ and ω^p, ω are linearly independent over $\mathbf{F}_p$.

4. Construction of Quantum Codes for p^m-State Systems from Classical Codes

4.1 Codes for p^m-State Systems

In this subsection we show a construction of quantum codes for p^m-state systems from classical linear codes over $\mathbf{F}_{p^{2m}}$. Our construction is based on the construction [4] by Chen which constructs quantum codes for 2-state systems from linear codes over $\mathbf{F}_{2^{2m}}$. We modify his construction so that we can estimate the minimum weight (4) from the original code over $\mathbf{F}_{p^{2m}}$.

We fix a normal basis $\{\theta, \theta^p, \ldots, \theta^{p^{2m-1}}\}$ of $\mathbf{F}_{p^{2m}}$ over $\mathbf{F}_p$. There always exists a normal basis of $\mathbf{F}_{p^{2m}}$ over $\mathbf{F}_p$ [11, Sect. VI,§13]. For $\boldsymbol{a} = (a_1, \ldots, a_m, b_1, \ldots, b_m), \boldsymbol{a}' = (a_1', \ldots, a_m', b_1', \ldots, b_m') \in \mathbf{F}_p^{2m}$, we define $\phi(\boldsymbol{a}) = a_1\theta + a_2\theta^p + \cdots + a_m\theta^{p^{m-1}} + b_1\theta^{p^m} + \cdots + b_m\theta^{p^{2m-1}}$, and $T(\boldsymbol{a}, \boldsymbol{a}') = c_{m+1} - c_1 \in \mathbf{F}_p$, where $\phi(\boldsymbol{a})\phi(\boldsymbol{a}')^{p^m} = c_1\theta + \cdots + c_{2m}\theta^{p^{2m-1}}$ and $c_i \in \mathbf{F}_p$. Then T is a bilinear form.

Lemma 9: T is alternating and nondegenerate.

Proof: First we show that T is alternating, that is, $T(\boldsymbol{a}, \boldsymbol{a}) = 0$ for all $\boldsymbol{a} \in \mathbf{F}_p^{2m}$. Let $x = \phi(\boldsymbol{a}) \in \mathbf{F}_{p^{2m}}$, and $xx^{p^m} = c_1\theta + \cdots + c_{2m}\theta^{p^{2m-1}}$ for $c_i \in \mathbf{F}_p$. Then $(xx^{p^m})^{p^m} = c_{m+1}\theta + c_{m+2}\theta^p + \cdots + c_{2m}\theta^{p^{m-1}} + c_1\theta^{p^m} + \cdots + c_m\theta^{p^{2m-1}}$. Since $(xx^{p^m})^{p^m} = x^{p^m}x$, $c_i = c_{i+m}$ for $i = 1, \ldots, m$. Hence $T(\boldsymbol{a}, \boldsymbol{a}) = 0$.

We assume that $x \neq 0$, which implies that $\boldsymbol{a} \neq 0$. Since $x(\theta/x^{p^m})^{p^m} = \theta^{p^m}$, $T(\boldsymbol{a}, \phi^{-1}(\theta/x^{p^m})) = 1$, which shows the nondegeneracy. □

Lemma 10: By abuse of notation, we denote by T the representation matrix of the bilinear form T with respect to the standard basis of $\mathbf{F}_p^{2m}$, that is, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{F}_p^{2m}$, we have $T(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} T \boldsymbol{b}^t$. Let I_m be the $m \times m$ unit matrix and $S = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$. There exists a nonsingular $2m \times 2m$ matrix D such that $DTD^t = S$.

Proof: See [11, Chapter XV] and use the previous lemma. □

For $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathbf{F}_{p^{2m}}^n$, let $(a_{i,1}, \ldots, a_{i,m}, b_{i,1}, \ldots, b_{i,m}) = \phi^{-1}(c_i)D^{-1} \in \mathbf{F}_p^{2m}$. We define $\Phi(\boldsymbol{c}) = (a_{1,1}, a_{1,2}, \ldots, a_{1,n}, a_{2,1}, \ldots, a_{n,m}, b_{1,1}, \ldots, b_{n,m})$. Then it is clear that the Hamming weight of $\boldsymbol{c}$ is equal to the weight (4) of $\Phi(\boldsymbol{c})$, since D is a nonsingular matrix. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{F}_{p^{2m}}^n$ we consider the inner product

$$\langle \boldsymbol{a}, \boldsymbol{b}^{p^m} \rangle, \tag{6}$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbf{F}_{p^{2m}}^n$.

Proposition 11: Let $C \subset \mathbf{F}_{p^{2m}}^n$ be a linear code over $\mathbf{F}_{p^{2m}}$, and C' the orthogonal space of C with respect to (6). Then the orthogonal space of $\Phi(C)$ with respect to (1) is $\Phi(C')$.

Proof: For $\boldsymbol{e} = (e_1, \ldots, e_n), \boldsymbol{e}' = (e'_1, \ldots, e'_n) \in \mathbf{F}_{p^{2m}}^n$, the inner product (1) of $\Phi(\boldsymbol{e}) = (a_{1,1}, \ldots, a_{n,m}, b_{1,1}, \ldots, b_{n,m})$ and $\Phi(\boldsymbol{e}') = (a'_{1,1}, \ldots, a'_{n,m}, b'_{1,1}, \ldots, b'_{n,m})$ is equal to

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m}&(a_{i,j}b'_{i,j} - a'_{i,j}b_{i,j})\\ &= \sum_{i=1}^{n} \phi^{-1}(e_i)D^{-1}S(D^{-1})^t\phi^{-1}(e'_i)^t\\ &= \sum_{i=1}^{n} T(\phi^{-1}(e_i), \phi^{-1}(e'_i)).\end{aligned}$$

If $e_i e_i'^{p^m} = c_1\theta + \cdots + c_{2m}\theta^{p^{2m-1}}$, then $T(\phi^{-1}(e_i), \phi^{-1}(e'_i)) = c_{m+1} - c_1$. Thus if $\langle \boldsymbol{e}, \boldsymbol{e}'^{p^m}\rangle = 0$ then the inner product (1) of $\Phi(\boldsymbol{e})$ and $\Phi(\boldsymbol{e}')$ is zero, which implies $\Phi(C')$ is contained in the orthogonal space of $\Phi(C)$ with respect to (1). Comparing their dimensions as $\mathbf{F}_p$-spaces we see that they are equal. □

Theorem 12: Let $C \subset \mathbf{F}_{p^{2m}}^n$ be an $[n, (n-k)/2]$ linear code over $\mathbf{F}_{p^{2m}}$, $C^{p^m} = \{\boldsymbol{x}^{p^m} \mid \boldsymbol{x} \in C\}$, and $(C^{p^m})^\perp$ the orthogonal space of C^{p^m} with respect to the standard inner product. Suppose that $C \subseteq (C^{p^m})^\perp$, and the minimum Hamming weight of $(C^{p^m})^\perp \setminus C$ is d. Then the minimum weight (4) of $\Phi(C)$ is d, and $\Phi(C)$ is self-orthogonal with respect to the inner product (1). Any eigenspace of $\Phi(C)$ is an $[[n, k, d]]_{p^m}$ quantum code.

4.2 Error Correction for p^m-State Systems

In this subsection we consider how to determine the error from measurements with quantum codes obtained via Theorem 12. We retain notations from Theorem 12. Suppose that $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_r$ is an $\mathbf{F}_{p^{2m}}$-basis of C.

Suppose that by a similar procedure to Sect. 2.2, the error is converted to a unitary matrix corresponding to $\Phi(\boldsymbol{e})$ for $\boldsymbol{e} \in \mathbf{F}_{p^{2m}}^n$. We fix a basis $\{\alpha_1, \ldots, \alpha_{2m}\}$ of $\mathbf{F}_{p^{2m}}$ over $\mathbf{F}_p$. Then $\mathbf{F}_p$-basis of $\Phi(C)$ is $\{\Phi(\alpha_j \boldsymbol{g}_i) \mid i = 1, \ldots, r, j = 1, \ldots, 2m\}$. First we shall show how to calculate $\langle \boldsymbol{e}, \boldsymbol{g}_i^{p^m}\rangle$ for each i. For $j = 1, \ldots, 2m$, let $(\boldsymbol{a}_j|\boldsymbol{b}_j) = \Phi(\alpha_j \boldsymbol{g}_i)$. As in Sect. 3.2, by the measurement outcomes we can know the inner product (1) of $\Phi(\boldsymbol{e})$ and $\Phi(\alpha_j \boldsymbol{g}_i)$, denoted by s_j, for $j = 1, \ldots, 2m$.

For $x = c_1\theta + \cdots + c_{2m}\theta^{p^{2m-1}} \in \mathbf{F}_{p^{2m}}$, $c_1, \ldots, c_{2m} \in \mathbf{F}_p$, we define $P(x) = c_{m+1} - c_1$. Then P is a nonzero $\mathbf{F}_p$-linear map. As discussed in the proof of Proposition 11, $s_j = P(\langle \boldsymbol{e}, \alpha_j^{p^m} \boldsymbol{g}_i^{p^m}\rangle) = P(\alpha_j^{p^m}\langle \boldsymbol{e}, \boldsymbol{g}_i^{p^m}\rangle)$. We define the map $P_{2m} : \mathbf{F}_{p^{2m}} \to \mathbf{F}_p^{2m}$, $x \mapsto (P(\alpha_1^{p^m} x), \ldots, P(\alpha_{2m}^{p^m} x))$. Then P_{2m} is an $\mathbf{F}_p$-linear map, and $P_{2m}(\langle \boldsymbol{e}, \boldsymbol{g}_i^{p^m}\rangle) = (s_1, \ldots, s_{2m})$. If P_{2m} is an isomorphism, then finding $\langle \boldsymbol{e}, \boldsymbol{g}_i^{p^m}\rangle$ from $(s_1, \ldots, s_{2m})$ is a trivial task, merely a matrix multiplication. We shall show that P_{2m} is an isomorphism.

Lemma 13: [11, Theorem 6.1, Chapter III] Let W be a $2m$-dimensional vector space over a field K with a basis $\{x_1, \ldots, x_{2m}\}$, and $\widehat{W}$ the dual of W, that is, the K-linear space consisting of linear maps from W to K. Then there exists a basis $\{f_1, \ldots, f_{2m}\}$ of $\widehat{W}$ such that $f_k(x_j) = \delta_{jk}$. $\{f_1, \ldots, f_{2m}\}$ is called the *dual basis.*

Lemma 14: There exist $\beta_1, \ldots, \beta_{2m} \in \mathbf{F}_{p^{2m}}$ such that $P(\alpha_j^{p^m}\beta_k) = \delta_{jk}$.

Proof: Notice that $\{\alpha_1^{p^m}, \ldots, \alpha_{2m}^{p^m}\}$ is an $\mathbf{F}_p$-basis of $\mathbf{F}_{p^{2m}}$. The dual space $\widehat{\mathbf{F}}_{p^{2m}}$ can be regarded as $\mathbf{F}_{p^{2m}}$-linear space by defining $xf : u \mapsto f(xu)$ for $x \in \mathbf{F}_{p^{2m}}$ and $f \in \widehat{\mathbf{F}}_{p^{2m}}$. Let $f_1, \ldots, f_{2m}$ be the dual basis of $\{\alpha_1^{p^m}, \ldots, \alpha_{2m}^{p^m}\}$. Since $\widehat{\mathbf{F}}_{p^{2m}}$ is one-dimensional $\mathbf{F}_{p^{2m}}$-linear space and $0 \neq P \in \widehat{\mathbf{F}}_{p^{2m}}$, f_k can be written as $\beta_k P$ for some $\beta_k \in \mathbf{F}_{p^{2m}}$. It is clear that $P(\alpha_j^{p^m}\beta_k) = \delta_{jk}$. □

Proposition 15: P_{2m} is an isomorphism.

Proof: It suffices to show that P_{2m} is surjective. For $(a_1, \ldots, a_{2m}) \in \mathbf{F}_p^{2m}$, $P_{2m}(a_1\beta_1 + \cdots + a_{2m}\beta_{2m}) = (a_1, \ldots, a_{2m})$, where β_k is as in the previous lemma. □

As in Sect. 3.2, the error $\boldsymbol{e}$ can be determined by a classical error-correcting algorithm for $(C^{p^m})^\perp$ from $\langle \boldsymbol{g}_1^{p^m}, \boldsymbol{e}\rangle, \ldots, \langle \boldsymbol{g}_r^{p^m}, \boldsymbol{e}\rangle$.

5. Notes on the Construction of Codes with Efficient Decoding Algorithms

It is desirable to have a systematic construction of quantum codes with efficient decoding algorithms. Calderbank et al. [3, Sect. V] showed a construction of cyclic linear quantum codes using the BCH bound for the minimum distance. With their construction we can correct errors up to the BCH bound using the Berlekamp-Massey algorithm.

If we use the Hartmann-Tzeng bound [7] or the restricted shift bound [14] then we get a better estimation of the minimum distance, and we can correct more errors using modified versions of the Feng-Rao decoding algorithm in [14, Theorem 6.8 and Remark 6.12]. The algorithms [14, Theorem 6.8 and Remark 6.12] correct errors up to the Hartmann-Tzeng bound or the restricted shift bound.

We cannot construct good cyclic codes of arbitrary code length. So we have to often puncture a cyclic code as in [3, Theorem 6 b)] to get a quantum code with efficient decoding algorithms. In classical error correction, we correct errors of a punctured code by applying an error-and-erasure decoding algorithm for the original code to the received word. But there is no (classical) received word in quantum error correction. So we decode a quantum punctured code as follows: Let $C' \subset \mathbf{F}_q^n$ be a (classical) linear code, $\boldsymbol{h}'_1, \ldots, \boldsymbol{h}'_r$ the rows of a check matrix for C', C the punctured code of C' obtained by discarding the first coordinate, $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_{r-1}$ the rows of a check matrix for C, and $0\boldsymbol{h}_i$ the concatenation of 0 and $\boldsymbol{h}_i$ for $i = 1, \ldots, r-1$. We can express $0\boldsymbol{h}_i$ as

$$0\boldsymbol{h}_i = \sum_{j=1}^{r} a_{ij}\boldsymbol{h}'_j,$$

where $a_{ij} \in \mathbf{F}_q$ [15, Lemma 10.1]. Suppose that an error $\boldsymbol{e} = (e_2, \ldots, e_n) \in \mathbf{F}_q^{n-1}$ occurs and that we have the syndrome $s_1 = \langle \boldsymbol{h}_1, \boldsymbol{e}\rangle, \ldots, s_{r-1} = \langle \boldsymbol{h}_{r-1}, \boldsymbol{e}\rangle$. We want to find $\boldsymbol{e}$ from $s_1, \ldots, s_{r-1}$ using an error-and-erasure decoding algorithm for C'. We can find $s'_1, \ldots, s'_r \in \mathbf{F}_q$ such that

$$s_i = \sum_{j=1}^{r} a_{ij}s'_j$$

for $i = 1, \ldots, r-1$. Then there exists $\boldsymbol{e}' = (e_1, \ldots, e_n) \in \mathbf{F}_q^n$ such that

$$\langle \boldsymbol{h}'_j, \boldsymbol{e}'\rangle = s'_j \text{ for } j = 1, \ldots, r, \tag{7}$$

because the condition (7) implies that $\langle 0\boldsymbol{h}_i, \boldsymbol{e}'\rangle = s_i$ for $i = 1, \ldots, r-1$. If we apply an error-and-erasure decoding algorithm to the syndrome $s'_1, \ldots, s'_r$ with the erasure in the first coordinate, then we find $\boldsymbol{e}'$.

Note that the algorithms [14, Theorem 6.8 and Remark 6.12] are error-only decoding algorithms but we can modify them to error-and-erasure algorithms along the same line as [17, Sect. VI].

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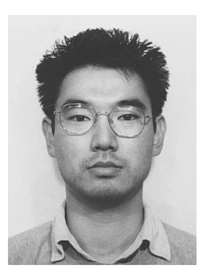

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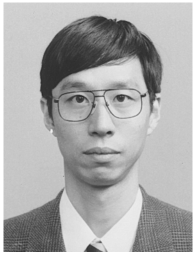

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